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BRIEF NOTE
NOISE-INDUCED EXTREMA IN TIME-DEPENDENT GINSBURG–LANDAU SYSTEMS

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Abstract—Qualitative changes in Ginsburg–Landau-type nonlinear systems are induced by simple alterations in the nature, not necessarily the amount, of additive noise.

A basic hypothesis of time-dependent Ginsburg–Landau theory, whether formulated in a Hamiltonian[1, 2] or in a Lagrangian[3] representation, is that the assumed additive microscopic noise is relatively independent of the nature of its source. Therefore stability and renormalization-group analyses have focussed on the behavior of extrema, disregarding possible influences by changes in the nature of the underlying noise on the nature or number of these extrema. In this paper it is demonstrated that even changes in the nature of additive noise in simple Ginsburg–Landau systems can increase the number of extrema that must be further analysed.

Consider the following two systems with a given Ginsburg–Landau type of nonlinearity, which are described by Langevin rate equations for two real variables $y^\alpha = \{y^1, y^2\}$ in the presence of additive noise arising from two microscopic sources $\xi^i = \{\xi^1, \xi^2\}$.

$$\dot{y}^\alpha = f^\alpha(y) + \upsilon_i^\alpha \xi^i. \tag{1}$$

To represent system I, take

$$\begin{aligned} f^1 &= y^1 - (y^2)^3, \\ f^2 &= -y^2 - (y^1)^3, \end{aligned} \tag{2a}$$

and to represent system II, take

$$\begin{aligned} f^1 &= -y^2 - (y^1)^3, \\ f^2 &= y^1 - (y^2)^3. \end{aligned} \tag{2b}$$

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The white noise is formally represented by

$$\langle \xi^i \rangle = 0,$$
$$\langle \xi^i(t)\xi^j(t') \rangle = \delta^{ij}\delta(t - t'),$$
$$\langle F(\xi) \rangle = \frac{\int d^2\xi F(\xi) \exp\left[-\frac{1}{2}\int_{t_0}^{\infty} d\tau\, \xi^i\xi^i\right]}{\int d^2\xi \exp\left[-\frac{1}{2}\int_{t_0}^{\infty} d\tau\, \xi^i\xi^i\right]}, \tag{3}$$

and where the summation convention is employed for all terms containing factors with repeated indices.

The importance of explicitly including the source of microscopic noise has been stressed in mathematical analyses of statistical mechanics[4, 5] and in physical applications[6–12]. In this study, one must at least appreciate that $\dim[\xi] \geq \dim[y]$. The results below depend only on the relative values of these microscopic sources, not their absolute values.

A path-integral Lagrangian L can be derived for these Langevin systems by deriving the conditional probability distribution P for y_t^α at time $t \equiv t_0 + (u + 1)\,\Delta t$, given y_0^α at initial time t_0, where u labels a prepoint-discretized representation[4, 5]:

$$P[y_t \mid y_0] = \lim_{u\to\infty} \lim_{\Delta t\to 0} \int_{y_0 = y_{t_0}}^{y_{u+1} = y_t} Dy \prod_{\rho=0}^{u} (2\pi\Delta t)^{-1} g^{1/2} \exp(-\Delta t L_\rho),$$
$$Dy = \prod_{\alpha}^{1,2} \prod_{\rho=1}^{u} dy_\rho^\alpha,$$
$$L = \tfrac{1}{2}(\dot{y}^\alpha - f^\alpha) g_{\alpha\beta} (\dot{y}^\beta - f^\beta), \tag{4}$$
$$\dot{y}_\rho^\alpha = (y_{\rho+1}^\alpha - y_\rho^\alpha)/\Delta t,$$
$$g = \det(g_{\alpha\beta}),$$

where the metric $g_{\alpha\beta}$ corresponding to systems I and II is derived from the variance $g^{\alpha\beta}$,

$$g^{\alpha\beta} = (g_{\alpha\beta})^{-1} = (\upsilon_i^\alpha \upsilon_i^\beta) = \begin{pmatrix} E^2 + \epsilon^2 & 2\epsilon E \\ 2\epsilon E & E^2 + \epsilon^2 \end{pmatrix}. \tag{5}$$

For this study, take $\upsilon_1^1 = \upsilon_2^2 = E$ and $\upsilon_2^1 = \upsilon_1^2 = \epsilon$, and it is assumed that $\det(g_{\alpha\beta}) \neq 0$, or $E \neq \epsilon$.

To explore extrema of the system, consider the "static" Lagrangian $\bar{L}$, setting $\dot{y}^\alpha =$ 0,

$$\bar{L} = \tfrac{1}{2} f^\alpha g_{\alpha\beta} f^\beta, \tag{6}$$

and the corresponding Euler–Lagrange equations,

$$\bar{L}_{,\gamma} = 0 = f^\alpha_{,\gamma} g_{\alpha\beta} f^\beta,$$
$$[\cdots]_{,\gamma} \equiv \partial[\cdots]/\partial y^\gamma. \tag{7}$$

Note that $\bar{L}$ is the same for both systems I and II, and that one set of solutions to $\bar{L}_{,\gamma} = 0$ is $f^\beta = 0$, the static solutions of Eq. (1). However, stability of this solution and of other solutions of Eq. (7) must still be examined[13]. This is not the issue here.

In system I for small y^α, it might be reasonable to assume $\epsilon = 0$, or an uncoupling with respect to noise. This might also be assumed for system II, for some physical models. Then

$$\bar{L} = \frac{(f^1)^2 + (f^2)^2}{2E^2}, \tag{8}$$

and the static Euler–Lagrange equations are independent of the noise.

However, in system I the noise may not be uncoupled. Also, perhaps more obvious when performing numerical integration of the differential equations, the noise in system II may not be uncoupled, as can be seen from a discretization of system II:

$$\begin{aligned} (y^1_{\rho+1} - y^1_\rho)/\Delta t &= -y^2_\rho - (y^1_\rho)^3 + \upsilon^1_i \xi^i_\rho, \\ (y^2_{\rho+1} - y^2_\rho)/\Delta t &= y^1_\rho - (y^2_\rho)^3 + \upsilon^2_i \xi^i_\rho. \end{aligned} \tag{9}$$

It is seen that the prediction of $y^\alpha_{\rho+1}$ may reasonably have comparable uncertainties contributed from y^1_ρ and y^2_ρ.

To consider the coupled noise case, define

$$E = r\epsilon, \tag{10}$$

where $r \notin \{0, 1, \infty\}$. To study the extrema of this case, it is convenient to examine planes in $(\bar{L} - y^\alpha)$-space perpendicular to the y^1–y^2 plane through the point $y^\alpha = 0$. For example, consider perpendicular planes defined by

$$z \equiv y^2 = my^1. \tag{11}$$

To further examine the region about $y^\alpha \simeq 0$, keep terms up to z^4 in $\bar{L}$ and obtain

$$\bar{L} \simeq \frac{z^2[1 - (r + r^{-1})m + m^2] + z^4[-(r + r^{-1}) + 2m - 2m^3 + (r + r^{-1})m^4]}{2r\epsilon^2(r + r^{-1})}. \tag{12}$$

It is straightforward to determine that for

$$1 < m < \max\{r, r^{-1}\} \tag{13}$$

the coefficient of z^2 in Eq. (12) is negative and the coefficient of z^4 is positive, leading to bifurcation.

Therefore it is demonstrated that for such systems as I and II, the extrema structure is altered by changing the degree of coupled noise represented by r. This analysis also demonstrates the utility of the path-integral representation in making more transparent some properties of stochastic nonlinear systems. These same qualitative results may be expected in other nonlinear stochastic systems, even with only additive noise. Multiplicative noise introduces even more structure into these systems[4, 5], as demonstrated for the statistical mechanics of neocortical interactions[7–11], for nonlinear nonequilibrium financial markets[6], for artificial intelligence[14] and for nuclear forces where the quantum analog of $g_{\alpha\beta}$ is nonconstant[15–17].

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